JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **35**, No. 1, February 2022
http://dx.doi.org/10.14403/jcms.2022.35.1.25

# WHICH WEIGHTED SHIFTS ARE M-HYPONORMAL?

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Abstract. Let $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ be a weight sequence and let $W_\alpha$ denote the associated unilateral weighted shift on $l^2(Z_+)$. In this paper we will investigate which weighted shift is M-hyponormal.

## 1. Introduction

Let $B(H)$ be the algebra of bounded linear operators on a separable complex Hilbert space H. An operator $T \in B(H)$ is called normal if $T^*T = TT^*$ and hyponormal if $T^*T \geq TT^*$. An operator $T \in B(H)$ is called M-hyponormal if there exists $M > 0$ such that

$$\| (T-\lambda)^* x \| \leq M \| (T-\lambda)x \| \text{ for all } \lambda \in \mathbb{C} \text{ and for all } x \in H.$$

Note that if T is an M-hyponormal operator then $M \geq 1$ and T is hyponomal iff M=1.
If $M \leq 1$ then M-hyponormality implies hyponormality. The notion of an M-hyponormal operator is due to Stampfli(unpublished)(see[9]).
It is an easy extension of hyponormal operators,

$$\text{T is hyponormal} \Rightarrow \text{T is M-hyponormal.}$$

Recall that given a bounded sequence of positive numbers $\alpha$: $\alpha_0$,$\alpha_1$,... (called weights), the unilateral weighted shift $W_\alpha$ associated with $\alpha$ is the operator on $l^2(Z_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis for $l^2(Z_+)$ (where $Z_+$ is the set of non-negative integers). We also simply write $W_\alpha \equiv$ shift$(\alpha_0, \alpha_1, ...)$. It is well known that $T \equiv W_\alpha$ is hyponormal but not normal if and only if $\alpha$ is monotonically increasing. Wadha[9] gave an example of an M-hoponormal operator which is not hyponormal. An example is : Let $\{e_i\}_{i=1}^{\infty}$ be an orthogonal basis of a Hilbert space H.

Received January 29, 2022; Accepted February 18, 2022.
2010 Mathematics Subject Classification: Primary 47B20.
Key words and phrases: M-hyponormal operator.

Let T be a weighted shift defined by T$e_1$=$e_2$, T$e_2$=2$e_3$ and T$e_i$=$e_{i+1}$ for $i \geq 3$. In paper [4], as a result of studying which weigfted shift is M-hyponormal, it was shown that if $\alpha$ is eventually increasing then $T \equiv W_\alpha$ is M-hyponormal. In this paper, we will try to find another case of which weighted shift becomes M-hyponormal.
On the other hand, Radjabalipour[6] showed that the only quasinilpotent M-hyponormal operator is 0. Thus if $W_\alpha$ is a weighted shift with weight sequence $\{\alpha_n\}$ converging to 0 then $W_\alpha$ is not M-hyponormal. In this paper, we are trying to prove the above fact directly by using the definition of M-hyponormality.

## 2. Preliminaries

In this section we give the definition of M-hyponormal operator and we shall state some general properties of M-hyponormal operator.

DEFINITION 2.1. An operator $T \in B(H)$ is called M-hyponormal if there exists a real number $M > 0$ such that

$$\| (T-\lambda)^* x \| \leq M \| (T-\lambda)x \| \text{ for all } \lambda \in \mathbb{C} \text{ and for all } x \in H.$$

The following facts follow from the above definition and some well known facts about M-hyponormal operators. The following results are proved by the definition of M-hyponormality or similar to hyponormal operators.

PROPOSITION 2.2. *T is an M-hyponormal operator iff*

$$M^2(T-\lambda)^*(T-\lambda) - (T-\lambda)(T-\lambda)^* \geq 0 \textit{ for all } \lambda \in C.$$

PROPOSITION 2.3. *If T is an M-hyponormal operator, then*
*(i)* $Tx = \lambda x$ *implies that* $T^*x = \bar{\lambda}x$ ;
*(ii)* $\| (T^* - \bar{\lambda})^{-1} x \| \leq M \| (T-\lambda)^{-1} x \|$ *for all* $\lambda$ *in resolvent set of T* ;
*(iii)* $\| (T-\lambda)x \|^{n+1} \leq M^{n(n+1)/2} \| (T-\lambda)^{n+1} x \|$.

PROPOSITION 2.4. *Let T be an M-hyponormal operator.*
*(i)* *If* $(T-\lambda)^n x = 0, then (T-\lambda)x = 0$.
*(ii)* *If* $Tx = \lambda_1 x$ *and* $Ty = \lambda_2 y$, $\lambda_1 \neq \lambda_2$, *then* $(x, y) = 0$.
*(iii) If there exists a polynomial* $p(\lambda)$ *such that* $p(T) = 0$,
*then T is normal.*
*(iv) If H is finite dimensional, then T is normal.*

The following result simplifies the construction of a dominant operator that is not M-hyponormal for any $M > 0$(see[6]).

PROPOSITION 2.5. *Every M-hyponormal quasinilpotent operator is zero. Therefore if* $\{e_n\}_{-\infty<n<\infty}$ *is an orthonormal basis for* $H$ *and if* $T$ *is the bilateral weighted shift defined by* $Te_n = 2^{-|n|}e_{n+1}$, *then both* $T$ *and* $T^*$ *are compact, quasinilpotent and dominant, but are not M-hyponormal for any* $M > 0$.

With the following result, we can see which weighted shift becomes the M-hyponormal(see[4]).

**Theorem** 2.6. *Let* $T \equiv W_\alpha$ *be a weighted shift with weight sequence* $\alpha = \{\alpha_n\}_{n=0}^{\infty}$. *If* $\alpha$ *is eventually increasing then* $T$ *is M-hyponormal.*

**Theorem** 2.7. *Let* $T \equiv W_\alpha$ *be a weighted shift with weight sequence* $\alpha = \{\alpha_n\}_{n=0}^{\infty}$. *If* $\alpha$ *has exactly two subsequential limits such that the larger one is different from the spectral radius* $r(T)$ *of* $T$, *then* $T$ *is not M-hyponormal.*

## 3. Every M-hyponormal quasinilpotent operator is zero

In this section, we are trying to prove the fact that "if $W_\alpha$ is a weighted shift with weight sequence $\{\alpha_n\}$ converging to 0 then $W_\alpha$ is not M-hyponormal" directly by using the definition of M-hyponormality. We try to solve the problem with an example and see if we can generalize it.

EXAMPLE 3.1. *Let* $T \equiv W_\alpha$ *be a weighted shift with weight sequence* $\alpha = \{\alpha_n\}_{n=0}^{\infty}$. *Let* $\alpha_n = \frac{1}{n+1}$. *Then* $T \equiv W_\alpha$ *is not M-hyponormal.*

*Proof.* Suppose that T is M-hyponormal. Then there exists $M \geq 0$ such that $M \parallel (T-\lambda)x \parallel \geq \parallel (T-\lambda)^*x \parallel$ for all $x \in H$ and for all $\lambda \in \mathbb{C}$.
Let $\lambda \neq 0$ and choose a sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_0 = 1 \text{ and } x_n = \frac{1}{\lambda^n}\prod_{j=0}^{n-1}\alpha_j \quad (n = 1, 2, ...).$$

By ratio test, if $\lambda \neq 0$ then $\sum_{n=0}^{\infty} x_n^2$ will converge. Thus $x = \sum_{n=0}^{\infty} x_n e_n \in$

$l^2$ for $\lambda \neq 0$. Then for all $x \in l^2$,

$$\begin{aligned}
&M^2\| (T-\lambda)x \|^2 - \| (T^*-\bar{\lambda})x \|^2 \\
=&(M^2-1)\| (T-\lambda)x \|^2 + \| (T-\lambda)x \|^2 - \| (T^*-\bar{\lambda})x \|^2 \\
=&(M^2-1)\{| \lambda x_0 |^2 + \sum_{i=0}^{\infty} | \alpha_i x_i - \lambda x_{i+1} |^2\} + | \alpha_0 x_0 |^2 + \sum_{i=1}^{\infty} ({\alpha_i}^2 - {\alpha_{i-1}}^2) | x_i |^2 \\
=&(M^2-1)| \lambda |^2 + 1 + \sum_{i=1}^{\infty} \left\{ \left(\frac{1}{i+1}\right)^2 - \left(\frac{1}{i}\right)^2 \right\} \frac{1}{(i!)^2 | \lambda |^{2i}}.
\end{aligned}$$

If $0 <| \lambda |^2 < \frac{\sqrt{3M^2-2}-1}{2(M^2-1)}$, then

$$\begin{aligned}
&M^2\| (T-\lambda)x \|^2 - \| (T^*-\bar{\lambda})x \|^2 \\
&= (M^2-1)| \lambda |^2 + 1 + \sum_{i=1}^{\infty} \left\{ \left(\frac{1}{i+1}\right)^2 - \left(\frac{1}{i}\right)^2 \right\} \frac{1}{(i!)^2 | \lambda |^{2i}} \\
&< (M^2-1)| \lambda |^2 + 1 - \frac{3}{4 | \lambda |^2} \\
&< 0.
\end{aligned}$$

Therefore T is not M-hyponormal. □

Example 3.2. *Let $T \equiv W_\alpha$ be a weighted shift with weight sequence $\alpha = \{\alpha_n\}_{n=0}^{\infty}$. Let $\{\alpha_n\}$ be decreasing sequence and* $\lim_{n\to\infty} \alpha_n = 0$. *Then $T \equiv W_\alpha$ is not M-hyponormal.*

*Proof.* Suppose that T is M-hyponormal. Then there exists $M \geq 0$ such that $M \| (T-\lambda)x \| \geq \| (T-\lambda)^* x \|$ for all $x \in H$ and for all $\lambda \in \mathbb{C}$.
Let $\lambda \neq 0$ and choose a sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_0 = 1 \text{ and } x_n = \frac{1}{\lambda^n} \prod_{j=0}^{n-1} \alpha_j \quad (n = 1, 2, ...).$$

By ratio test, if $\lambda \neq 0$ then $\sum_{n=0}^{\infty} x_n^2$ will converge. Thus $x = \sum_{n=0}^{\infty} x_n e_n \in$

$l^2$ for $\lambda \neq 0$. Then for all $x \in l^2$,

$$
\begin{aligned}
&M^2 \| (T-\lambda)x \|^2 - \| (T^* - \bar{\lambda})x \|^2 \\
=&(M^2-1)\| (T-\lambda)x \|^2 + \| (T-\lambda)x \|^2 - \| (T^* - \bar{\lambda})x \|^2 \\
=&(M^2-1)\{| \lambda x_0 |^2 + \sum_{i=0}^{\infty} | \alpha_i x_i - \lambda x_{i+1} |^2\} + | \alpha_0 x_0 |^2 + \sum_{i=1}^{\infty} ({\alpha_i}^2 - {\alpha_{i-1}}^2) | x_i |^2 \\
=&(M^2-1)| \lambda |^2 + {\alpha_0}^2 + \sum_{i=1}^{\infty} ({\alpha_i}^2 - {\alpha_{i-1}}^2) | x_i |^2 .
\end{aligned}
$$

If $0 <| \lambda |^2 < \frac{\sqrt{4(M^2-1)({\alpha_0}^2 - {\alpha_1}^2){\alpha_0}^2 + {\alpha_0}^4} - {\alpha_0}^2}{2(M^2-1)}$, then

$$
\begin{aligned}
&M^2 \| (T-\lambda)x \|^2 - \| (T^* - \bar{\lambda})x \|^2 \\
=&(M^2-1)| \lambda |^2 + {\alpha_0}^2 - \sum_{i=1}^{\infty} ({\alpha_{i-1}}^2 - {\alpha_i}^2) | x_i |^2 \\
<&(M^2-1)| \lambda |^2 + {\alpha_0}^2 - ({\alpha_0}^2 - {\alpha_1}^2) | x_1 |^2 \\
<&(M^2-1)| \lambda |^2 + {\alpha_0}^2 - ({\alpha_0}^2 - {\alpha_1}^2) \frac{{\alpha_0}^2}{| \lambda |^2} \\
<&0.
\end{aligned}
$$

Therefore T is not M-hyponormal. □

## 4. Which weighted shifts are M-hyponormal?

In paper [4], it was shown that if $\alpha$ is eventually increasing then $T \equiv W_\alpha$ is M-hyponormal. Here, we define sequences of bounded variation, essentially increasing, or, essentially decreasing sequences and try to use them to examine which weighted shift is M-hyponormal.

Definition 4.1. The sequence $a_1$, $a_2$, $a_3$,...
of real or complex numbers is said to be of 2-bounded variation
iff it satisfies

$$\sum_{n=1}^{\infty} | {a_n}^2 - {a_{n+1}}^2 | < \infty.$$

**Theorem** 4.2. *If* $\sum_{n=1}^{\infty} | {a_n}^2 - {a_{n+1}}^2 | < \infty$, *then* $(a_n)$ *is convergent.*

*Proof.* Let a sequence $(a_n)$ be of 2-bounded variation. Then for $m < n$,

$$a_m{}^2 - a_n{}^2 = \sum_{i=m}^{n-1}(a_i{}^2 - a_{i+1}{}^2).$$

Then

$$\mid a_m{}^2 - a_n{}^2 \mid \leq \sum_{i=m}^{n-1} \mid a_i{}^2 - a_{i+1}{}^2 \mid .$$

By the Cauchy Criterion for convergence of series, the sequence $(a_n)$ is a Cauchy sequence and thus converges. □

DEFINITION 4.3. A sequence $(a_n)_{n\geq 1}$ is called an essentially increasing sequence if
(i)$\exists \lim_{n\to\infty} a_n = a$
(ii)there exists N such that $a_n \leq a$ for all $n \geq N$
and $(a_n)_{n\geq 1}$ is called an essentially decreasing sequence if
(i)$\exists \lim_{n\to\infty} a_n = a$
(ii)there exists N such that $a_n \geq a$ for all $n \geq N$.

EXAMPLE 4.4. *Let $T \equiv W_\alpha$ be a weighted shift with weight sequence $\alpha = \{\alpha_n\}_{n=0}^{\infty}$. Let*

$$\alpha_n = \begin{cases} \sqrt{1 - \frac{1}{k}} & \text{if } n = 2k-5, k = 3,4,5,... \\ \sqrt{1 - \frac{2}{k}} & \text{if } n = 2k-6, k = 3,4,5,... \end{cases}$$

*Then $\alpha$ is an essentially increasing sequence,*
*but $\sum_{n=1}^{\infty} \mid a_n{}^2 - a_{n+1}{}^2 \mid = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$.*
*By the above facts, $\alpha$ is an essentially increasing sequence but is not to be of 2-bounded variation. So the sequence of 2-bouned variation condition is required.*

The conjecture using the two definition is as follows:

CONJECTURE 4.5. *Let $T \equiv W_\alpha$ be a weighted shift with weight sequence $\alpha = \{\alpha_n\}_{n=0}^{\infty}$. If $\alpha$ is an essentially increasing sequence and $\sum_{i=1}^{\infty} \mid \alpha_i{}^2 - \alpha_{i-1}{}^2 \mid < \infty$ , then $T \equiv W_\alpha$ is M-hyponormal.*

We have tried hard to prove it, but we haven't been able to complete the proof yet. If this is proven, the converse of Theorem 1 in paper[4] does not hold.

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